

Scattering matrices and Affine Hecke Algebras.

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These lectures deal with two related topics. The first one is the construction of the scattering matrix for an arbitrary Weyl group. The second topic is the application of the affine Hecke algebra to construct physical models with enlarged symmetries.

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1. introduction

These lectures deal with two related topics.

The first one is the construction of the scattering matrix for an arbitrary Weyl group. The aim is to define a set of commuting operators in terms of generators which obey the Yang-Baxter equation. These operators which coincide with the usual scattering matrices form a group isomorphic to the weight lattice of the corresponding Lie group. In the first lecture, we interpret the Yang Baxter generators as elementary reflections. This geometrical picture enables us to construct the scattering matrices in a systematic way and to show that they commute.

In this lecture, we have used the language of Coxeter-Weyl groups and cells exposed in [1], [2].

The second topic is the application of the affine Hecke algebra to construct physical models with enlarged symmetries.

The affine Hecke algebra is a deformation of the affine Weyl groups. The reflections across the origin are deformed into the none affine generators of the Hecke algebra and the translations are deformed into generators which coincide with the scattering matrices of the first lecture. Using a specific representation of the Yang Baxter generators which uses the Hecke algebra, we show that the scattering matrices obey the affine Hecke relations. We then derive a representation in a space of polynomials in several variables. This enables us to construct a set of commuting Hamiltonians essentially given by the center of the affine Hecke algebra. These Hamiltonians are Hermitian and their spectrum can be computed in a simple way. One of their main interests is that they commute with q-deformations of the usual affine Lie algebras defined in terms of quadratic algebras.

The abstract theory of affine Hecke algebras is exposed in [3]. We expand here the point of view of [4], [5]. Parallel arguments are in [6] and some recent results can be found in [7].

2. The scattering matrix

Two important tools of soluble models are the transfer matrix an the scattering matrix. In his study of the delta interacting gas, Yang introduced a commuting family of scattering matrices. He obtained the spectrum by diagonalizing simultaneously these matrices. A simple way to prove the commutation relations for the scattering matrices consisted in defining a commuting family of transfer matrices depending of a continuous parameter.

By letting the parameter take some special values, he recovered the scattering matrices. This approach is expanded in the book of M.Gaudin [8].

I do not know a generalization of the transfer matrix for an arbitrary Weyl group, and when this can be done, (A,B,C,D cases), one cannot simply obtain the scattering matrix by letting the spectral parameter take special values except in the A case. On the other hand, the scattering matrix can be easily generalized to arbitrary Weyl groups. If a proper correspondence is used, the scattering matrices are identified with translations in the affine Weyl group. To generalize the construction of the scattering matrices it is useful to give them a geometrical interpretation. It enables to construct commuting operators from a set of generators obeying the (generalized) Yang-Baxter algebra.

In the first part, I recall some known facts about Weyl groups following the presentation of Carter [1]. I modify the presentation of the Weyl group in terms of generators and relations to introduce generators which obey the Yang-Baxter equation. The difference between the two approaches is analogous to the description of a rigid body motion in the moving frame compared to the description in the rest frame.

In the second part, I extend these considerations to the affine case which enables to obtain the expression of the scattering matrix. The geometrical interpretation is the following: Periodic trajectories inside the fundamental cell associated to a Weyl group are in correspondence with translations in the weight lattice. To these trajectories, we associate operators by taking the time ordered product of the Yang-Baxter operators of the reflections across the walls hit by the trajectory. These operators form a commutative group isomorphic to the weight lattice.

2.1. Systems of roots and Weyl groups

Root systems

Let V be an euclidian space of dimension l . for r a vector of v we denote by w_r the reflection in the hyperplane orthogonal to r .

$$w_r(x) = x - 2 \frac{(r, x)}{(r, r)} r. \quad (2.1)$$

A system of roots is defined by a set Φ of non zero vectors spanning V which obey the following properties:

- 1) if $r, s \in \Phi$ then, $w_r(s) \in \Phi$.
- 2) if $r, s \in \Phi$ then $2(r, s)/(r, r)$ is a rational integer.

3) if $r, \lambda r \in \Phi$ then $\lambda = \pm 1$

It will be convenient to denote $r^V = 2r/(r, r)$. The lattice spanned by the r^V is denoted Q^V .

We shall also consider the lattice spanned by the vectors p such that (p, r) is a rational integer for any root $r \in \Phi$. This lattice denoted P^V contains the coroot lattice Q^V .

One can define a basis of V , Π , which is a subset of Φ and is such that any root of Φ is a linear combination of roots of Π with coefficients which are either all non-negative or all non-positive. Π is called a fundamental system of roots. $\Pi = \{r_1, r_2, \dots, r_l\}$ and a root can be written:

$$r = \sum_{i=1}^l \lambda_i r_i \quad (2.2)$$

where either $\lambda_i \geq 0$ for all i , or $\lambda_i \leq 0$ for all i . Accordingly, we say that r is a positive $r \in \Phi^+$ or a negative root $r \in \Phi^-$. We say that $r > s$ if $r - s$ is a linear combination with coefficients ≥ 0 of elements in Π . Every positive systems of roots contains just one fundamental system. Thus there is a one to one correspondence between positive systems and fundamental systems in Φ .

chambers

For our purpose, it is useful to have a geometric interpretation of a fundamental system. For each root $r \in \Phi$, we denote by H_r the hyperplane orthogonal to r . The set $V - \cup H_r$ is disconnected, its connected components are called chambers. The roots orthogonal to the bounding hyperplanes of a chamber and pointing into the chamber form a fundamental system Π . A chamber C_Π is thus defined as the set of points $x \in V$ such that:

$$(r_i, x) > 0, \forall r_i \in \Pi \quad (2.3)$$

Moreover, every fundamental system arises in this way from some chamber.

Let C be a chamber, and $\delta(C)$ be its boundary. The bounding hyperplanes of C are called the walls of C and their intersection with $\delta(C)$ the faces of C . Two chambers which have a face in common are called adjacent.

Weyl group

The group W generated by the reflections w_r is called the Weyl group of Φ . Each element of W transforms Φ into itself. One can show that the Weyl group is generated by the fundamental reflections w_r with $r \in \Pi$. Given two chambers C and C' (alternatively two fundamental systems Π and Π'), there is a unique element of the Weyl group such

that $w(C) = C'$. Let us here give a construction of this element. This presentation will make the Yang-Baxter equation appear naturally in this context.

We consider a sequence C_0, C_1, \dots, C_m of chambers with $C_0 = C$, $C_m = C'$ such that two consecutive chambers are adjacent. Each chamber is characterized by a fundamental system Π_i , and there is a unique $r_i \in \Pi_i$ such that $w_{r_i}C_i = C_{i+1}$. We define the element of the Weyl group such that $w(C) = C'$ by:

$$w = w_{r_{m-1}} w_{r_{m-2}} \dots w_{r_0} \quad (2.4)$$

length of the Weyl group elements

The length of the element w , $l(w)$ can be defined in two different ways. The first one being:

a) $l(w)$ is the number of positive roots turned by w into negative roots.

Equivalently, if we denote Φ^+ , Φ'^+ the positive roots respectively associated to the fundamental systems of C and C' :

$$l(w) = |\Phi^+ \cap \Phi'^-| \quad (2.5)$$

Let $r \in \Pi'$. Consider the word $w_r w$, since w_r changes the sign of only $\pm r$ in Φ' , one has in particular:

$$\begin{aligned} l(w_r w) &= l(w) + 1 \quad \text{if } r \in \Phi^+ \\ l(w_r w) &= l(w) - 1 \quad \text{if } r \in \Phi^- \end{aligned} \quad (2.6)$$

Now, consider an expression of the form (2.4) to represent the Weyl group element w . From (2.6) it follows that the minimal number of reflections in (2.4) is larger than $l(w)$. Let us assume that the number of terms is strictly larger than $l(w)$. This means that the expression contains a reflection w_{-r} with $r \in \Phi^+$. From the above argument, this is only possible if the reflection w_r has occurred earlier. Thus, the expression is of the form:

$$w = w_{r_{m-1}} \dots w_{-r} X w_r \dots w_{r_0} \quad (2.7)$$

Let us denote $w_r(X)$ the expression obtained from X by substituting $w_{w_r(p)}$ for w_p in the expression of X . A shorter expression for w is given by:

$$w = w_{r_{m-1}} \dots w_r(X) \dots w_{r_0} \quad (2.8)$$

It follows that another definition of the length is:

b) $l(w)$ is the minimal length of an expression w in the form (2.4).

A word of minimal length is one for which all the r_i in the expression (2.4) of w are in $\Phi^+ \cap \Phi'^-$.

Given two chambers C and C' , let us describe the way to construct an element w of minimal length such that $w(C) = C'$. For this, we proceed by induction on l . If $l = 0$, $C = C'$ and $w = 1$. If $l \neq 0$, $\Phi'^- \cap \Pi$ is not empty. Let r_0 belong to this set. We put $w_{r_0}(C) = C_1$ and $\Pi_1 = r_0\Pi$. The set of positive roots defined by Π_1 is obtained from Φ by replacing r_0 by $-r_0$. Hence

$$|\Phi_1^+ \cap \Phi'^-| = l - 1 \quad (2.9)$$

and we can continue the construction replacing C by C_1 .

Yang-Baxter equation

We now give a presentation of the Weyl group W as an abstract group in terms of generators x_r , $r \in \Phi$ using the above construction. Namely, we consider the elements (2.4) with the reflection w_r replaced by x_r and we impose relations to the x_r in such a way that two elements w_1 and w_2 coincide whenever the Weyl group reflections are equal.

The operators x_r are subject to the following relations:

a) unitarity:

$$x_{-r}x_r = 1 \quad (2.10)$$

b) Generalized Yang-Baxter equation:

Let r and s be two roots in some fundamental system Π , and m_{rs} be the order of $w_r w_s$. Then, we have:

$$x_{r_1} x_{r_2} x_{r_3} \dots x_{r_m} = x_{r_m} x_{r_{m-1}} x_{r_{m-3}} \dots x_{r_1} \quad (2.11)$$

where : $r_1 = s$, $r_2 = w_s(r)$, $r_3 = w_s w_r(s)$, ..., $r_m = r$

For example, in the case where the root system is A_n , if we take $r = e_1 - e_2$, $s = e_2 - e_3$, then $m_{rs} = 3$ and the above equation writes:

$$x_{e_2 - e_3} x_{e_1 - e_3} x_{e_1 - e_2} = x_{e_1 - e_2} x_{e_1 - e_3} x_{e_2 - e_3} \quad (2.12)$$

one recognizes the usual Yang-Baxter equation. These equations are generalizations of the Yang-Baxter equation for arbitrary Weyl groups.

2.2. Affine root systems and Weyl groups

affine-root systems

In order to define a scattering matrix, we need to extend the previous considerations to the affine case.

We denote by E_0 an affine vector space of dimension d . For $r \in \Phi$ and $k \in \mathbb{Z}$ we define a hyperplane in E_0 by:

$$H_{r,k} = \{x \in E_0 | (r, x) = k\} \quad (2.13)$$

The orthogonal reflections with respect to the Hyperplanes $H_{r,k}$ generate a group called the affine Weyl group. This group is also the semi direct product of the Weyl group defined earlier and the translations in the coroot lattice Q^V .

The affine-roots are defined by a couple of non affine root and a rational integer. We denote $\tilde{\Phi}$ the set of affine roots. Each affine root $r + k\delta$ defines the Hyperplane $H_{r,k}$ and the corresponding reflection in the affine Weyl group:

$$w_{r+k\delta}(x) = x - ((r, x) - k)r^V \quad (2.14)$$

A fundamental system $\tilde{\Pi}$ is given by a basis of $\tilde{\Phi}$ such that any root of $\tilde{\Pi}$ is a linear combination of roots of $\tilde{\Pi}$ with coefficients which are either all positive ($\tilde{\Phi}^+$) or all negative ($\tilde{\Phi}^-$). For example, given a fundamental system Π , if we denote r_m is the largest root of Φ , we obtain a fundamental system $\tilde{\Pi}$ given by:

$$\tilde{\Pi} = \{r | r \in \Pi\} \cup \{-r_m + \delta\} \quad (2.15)$$

The positive roots are given by:

$$\tilde{\Phi}^+ = \{r + k\delta | r \in \Phi, k > 0\} \cup \{r | r \in \Phi^+\} \quad (2.16)$$

Cells

The set $E_0 - \cup H_{r,k}$ is disconnected, its connected components are called cells. A cell $C_{\tilde{\Pi}}$, $\tilde{\Pi} = \{r_i + k_i\delta, i = 0, \dots, r\}$, is characterized by the fundamental system given by the roots which define its bounding hyperplanes and which point into it:

$$C_{\tilde{\Pi}} = \{x \in E_0 | (r_i, x) \geq k_i, \forall i\} \quad (2.17)$$

The faces of a cell are defined as earlier, and two cells which have a face in common are called adjacent.

representation of the Weyl group elements by words

We denote by E_p the affine vector space centered at some point p . $E_0 = \{0, x\}$, $E_p = \{p, x\}$, with the identification $\{0, x\} = \{p, x - p\}$. To simplify the notations, $\{0, x\}$ is denoted x . In the following, we shall restrict to the affine spaces E_p with p in the coweight lattice: $p \in P^V$.

Let us define operators which intertwine E_p and E_{p+a} and which are equal to one when we use the equivalence relation to identify the two spaces. They simply translate the origin of the affine space. They are defined by:

$$t^a \{p, x\} = \{p + a, x - a\}$$

We also consider the affine reflections x_r for $r \in \Phi$ acting in E_p as:

$$x_r \{p, x\} = \{p, w_r(x)\}$$

Inside the parentheses, w_r denotes the Weyl reflection defined earlier. Using the equivalence relation to identify E_p with E_0 , the reflection $x_r|_{E_p}$ is identified with $w_{r+(r,p)\delta}$.

Given two cells, C and C' , there is a unique element w of the affine Weyl group which maps the first one into the second one. Let us here construct a word representing this element using the elementary transformations x_r and t^a . We consider a sequence of cells $C_0 = C, C_1, \dots, C_m = C'$ such that two consecutive cells are adjacent. To simplify the discussion, we fix C to be defined by the fundamental system (2.15). We consider a linear transformation acting in E_0 as follow:

$$S = t^{a_m} x_{r_{m-1}} t^{a_{m-1}} x_{r_{m-2}} \dots t^{a_1} x_{r_0} t^{a_0} \quad (2.18)$$

The reflection x_{r_i} acts in $E_{a_i+a_{i-1}+\dots}$ and maps the cell C_i onto the cell C_{i+1} . The origin $a_i + a_{i-1} + \dots$ must therefore belong to the common wall of the cell C_i and C_{i+1} . The root r_i is orthogonal to this wall and we take it to point into the cell C_i . It is clear that S intertwines E_0 with E_p where: $p = a_0 + a_1 + \dots + a_m$.

Using the equivalence relation to identify E_0 with E_p , one can view the transformation S in the affine Weyl group. Conversely any element w in the affine Weyl group is characterized by a cell C' and can be represented by a word (2.18). We can always choose $p \in Q^V \cap C'$, in this way w characterizes a unique point $p \in Q^V$ which is a summit of C' .

In this case, we can identify the Weyl group element w with the word denoted S_w which represents it.

Here, we need to be more general and we take $p \in P^V \cap C'$. Unless $P^V = Q^V$, this does not determine a unique point p . In that case, we denote by $S_{w,p}$ the word (2.18) which intertwines E_0 and E_p and which coincides with the element w in the affine Weyl group. p is a summit of C' and can be written in a unique way as

$$p = q - \gamma \quad (2.19)$$

where $q \in Q^V \cap C'$.

Scattering matrix

Let us restrict to the case where C' is obtained from C by a translation :

$$C' = C + \xi \quad (2.20)$$

In order for C' to define a cell, ξ must be in P^V . We denote by w_ξ the corresponding element in the affine Weyl group. (w_ξ is not a translation unless ξ is in the coroot lattice Q^V). In the construction of $S_{w_\xi,p}$, one can choose $p = \xi$. We denote by S_ξ the corresponding word $S_\xi = S_{w_\xi,\xi}$

In this case, γ defined in (2.19) must be a summit of C ; therefore, either $\gamma = 0$, or $\gamma \in P^V$ and $(\gamma, r_m) = 1$ where r_m is the largest root of Φ . Such weights are called minuscule weights and are representative of P^V/Q^V in the weight lattice.

Let us now consider that the sequence of cells C_i are defined up to a translation. We can view S_ξ as intertwining E_p and $E_{p+\xi}$ for $p \in P^V$ arbitrary. Then, it is clear that two words S_ξ and $S_{\xi'}$ can be multiplied and one has:

$$S_{\xi_1} S_{\xi_2} = S_{\xi_1 + \xi_2} \quad (2.21)$$

Thus these linear transformations form a commutative group isomorphic to the translations in the coweight lattice. Note that the above relation is not true for the Weyl group elements: $w_{\xi_1} w_{\xi_2} \neq w_{\xi_1 + \xi_2}$ unless ξ_2 is in the root lattice.

remark

We can be slightly more general and define a semi group multiplication law on the words $S_{C_1, p_1}^{C_2, p_2}$ which transform C_1, p_1 into C_2, p_2 by defining the product:

$$S_{C_2, p_2}^{C_3, p_3} S_{C_1, p_1}^{C_2, p_2} = S_{C_1, p_1}^{C_3, p_3} \quad (2.22)$$

length

We can define the length of a Weyl group element w as the minimal length of S_w in terms of operators x_r . The length is also given by the expression which generalizes (2.5),(2.6) in an obvious way.

$$l(w) = |\tilde{\Phi}^+ \cap \tilde{\Phi}'^-| \quad (2.23)$$

where $\tilde{\Phi}'^- = w(\tilde{\Phi}^-)$. The same definition applies to the generalized words $S_{w,p}$ since p is not relevant in the definition of the length.

In the case where the word (2.18) is of minimal length, one has:

$$\tilde{\Phi}^+ \cap \tilde{\Phi}'^- = \{r_i + \delta \sum_{j=0}^{i-1} (r_i, a_j)\} \quad (2.24)$$

Consider the dominant weights defined by:

$$P_{\text{dom}}^V = \{\xi \in P^V | (\xi, r) \geq 0, \forall r \in \Pi\} \quad (2.25)$$

Let us restrict to the words S_ξ where ξ is a dominant weight: $\xi \in P_{\text{dom}}^V$. In this case, it is not difficult to see that:

$$\tilde{\Phi}^+ \cap \tilde{\Phi}'^- = \{r + \delta k | r \in \Phi^+, 0 \leq k < (\xi, r)\} \quad (2.26)$$

and we have:

$$l(w_\xi) = \sum_{r \in \Phi^+} (\xi, r) \quad (2.27)$$

Moreover, it follows from (2.24) that the reduced expression of S_ξ , the operator x_r occurs exactly (r, ξ) times and the operator x_{-r} never occurs.

It is also straightforward to construct a word of minimal length by extending the method used in the non affine case. A geometrical interpretation of this construction is the following: One draws a straight line joining an arbitrary point A in the cell C to its translated $A' = A + \xi$ in C' . Traveling along this geodesic one goes successively through the cells C_i . The word S_ξ is obtained taking the product (from the right to the left) of the reflections across the hyperplanes which are successively come through by the trajectory. The translations t^a must be inserted in order for the origin of the affine space to belong to the reflecting hyperplane and one must have $\sum_{i=0}^m a_i = \xi$.

definition of words by generators and relations

We now proceed as before and replace the reflection operators x_r and the translation operators t^a by abstract operators. We impose relations to these operators so that the two words $S_{w,p}$ and $S_{w',p'}$ are equal when:

- a) They both intertwine E_0 with the same affine space E_p .
- b) After identifying E_0 with E_p , the two affine Weyl group elements w and w' coincide.

For this we impose the conditions (2.10) and (2.11) to the operators x_r and for the operators t^a we require:

- 1) that they form an additive group:

$$t^a t^b = t^{a+b} \quad (2.28)$$

- 2) that t^a commutes with x_r whenever r and a are orthogonal:

$$t^a x_r = x_r t^a, \text{ if } (a, r) = 0 \quad (2.29)$$

The elements S_ξ constructed in that way clearly commute, we call them scattering matrices. Using this construction we have obtained a method to construct a group of commuting operators isomorphic to the coweight lattice.

example:

Let us for example consider the case where the Weyl group is A_n . If we take the vector $\xi = -e_i$, the scattering matrix is given by:

$$S_{-e_i} = x_{e_i - e_{i-1}} x_{e_i - e_{i-2}} \dots x_{e_i - e_1} t^{-e_i} x_{e_i - e_n} \dots x_{e_i - e_{i+1}} \quad (2.30)$$

In the case where the t^a are equal to 1, these coincide with the scattering matrices considered by Yang.

In the next section, we shall use these transfer matrices to construct the q analogue of the Dunkl operators and we shall show that they obey the defining relations of a affine Hecke algebra.

3. The affine Hecke Algebra

In this section, we use the previous formalism to construct some representations of the affine Hecke algebra. In the first part, we recall some well known results about representations of the Yang-Baxter algebra using generators which obey the (non affine) Hecke algebra relations. This gives a certain representation of the scattering matrices. In the second part, we consider a limiting form of these scattering matrices to obtain a representation of the affine Hecke algebra. In the third part, we derive a representation of the Hecke and of the affine Hecke algebra acting in the group algebra of the weight lattice. We obtain in this way the q-Dunkl operators.

3.1. representation of the Yang-Baxter operators

Hecke algebra

Let us consider a set of operators g_r indexed by the roots $r \in \Phi$ on which the Weyl group acts in a natural way:

$$w_r g_s = g_{w_r(s)} w_r \quad (3.1)$$

and which obey the following relations:

a) Braid group relations:

Let r and s be two fundamental roots in Π , and $m_{r,s}$ be the order of $w_r w_s$, then:

$$g_r g_s \dots = g_s g_r \dots, \text{ } m_{r,s} \text{ terms on either side} \quad (3.2)$$

b) Hecke relations

$$(g_r - q_r)(g_r + q_r^{-1}) = 0 \quad (3.3)$$

where q_r is a complex number which depends on length of the root r .

The subset g_r with $r \in \Pi$ generates the Hecke algebra associated to Φ defined by:

$$g_w g_{w'} = g_{ww'} \text{ if } l(ww') = l(w) + l(w') \quad (3.4)$$

Note that we have $g_{w_r} = g_r$. only for $r \in \Pi$.

Group algebra of the weight lattice

The the group algebra of the Weight lattice is denoted P and is called the space of spectral parameters:

$$P = \{\sum c_\lambda e^\lambda | \lambda \in P^V\} \quad (3.5)$$

with the product given by:

$$e^\mu e^\lambda = e^{\lambda+\mu} \quad (3.6)$$

The Weyl group is not acting on P : $w_r e^\lambda = e^\lambda w_r$.

Yang-Baxter generators

It is straightforward to verify that the following operators obey the unitarity and the Yang-Baxter equation (2.10),(2.11):

$$x_r = w_r \frac{e^r g_r - g_r^{-1}}{e^r q_r - q_r^{-1}} \quad (3.7)$$

relations satisfied by the t^a :

In order for the operators t^a to obey the relations (2.29), we require that they commute with P and satisfy the following relations:

$$\begin{aligned} wt^a &= t^{w(a)}w, \quad \forall w \in W \\ t^a g_r &= g_r t^a, \quad \text{if } (r, a) = 0 \end{aligned} \tag{3.8}$$

Remarks:

a) Another way to realize (2.29) consists in having the operators t^a commute with the Weyl group and obey the following relation with P :

$$t^a e^r = e^{(r,a)} e^r t^a \tag{3.9}$$

such a realization occurs in the so called quantum Kniznik Zamolodchikov equation.

b) Let us consider the case of a root system for which the third condition defining a root system is not satisfied (nonreduced system) and there are proportional roots r and $2r$. Those are the BC systems for which the roots are: $\pm e_i$, $\pm 2e_i$, $\pm e_i \pm e_j$. In this case, one can find an expression for x_r which interpolates between the two forms of (3.7) for x_r and x_{2r} .

$$x_r = w_r \frac{e^{2r} g_r - g_r^{-1} + e^r \alpha}{e^{2r} q_r - q_r^{-1} + e^r \alpha} \tag{3.10}$$

This formula coincides with x_{2r} in (3.7) if $\alpha = 0$ and $q_{2r} = q_r$. It coincides with x_r if $\alpha = q_r - q_r^{-1}$. It also satisfies the unitarity condition (2.28) and the Yang-Baxter equation (2.29).

3.2. affine Hecke relations

In this section, we consider the Hecke algebra (3.4). We complete it by adding a group of translations isomorphic to the coweight lattice generated by the S_ξ . Moreover, we require that the translations and the generators g_r obey the commutation relations which define an affine-Hecke algebra. In addition to the relations (3.4) satisfied by the g_r , $r \in \Pi$ and the additive group formed by the S_ξ , $\xi \in P^V$, we require that the g_r and the S_ξ obey the relations:

Let $r \in \Pi$, and $\xi \in P^V$, then:

- i) if $(r, \xi) = 0$, $g_r S_\xi = S_\xi g_r$
- ii) if $(r, \xi) = 1$, $g_r S_\xi = S_{w_r(\xi)} g_r^{-1}$

Here we consider a limit of the operators S_ξ constructed in (2.18) which obey these relations. We consider the following limiting form (obtained when we set $e^r = \infty$ for $r > 0$ with respect a fixed fundamental system Π) of the operators x_r (3.7) :

$$\begin{aligned} x_r &= w_r g_r \text{ if } r \in \Phi^+ \\ x_r &= w_r g_r^{-1} \text{ if } r \in \Phi^- \end{aligned} \quad (3.12)$$

Let us first show these relations in the case where ξ is a dominant weight, $\xi \in P_{dom}^V$. We recall that in this case, S_ξ has a reduced expression:

$$S_\xi = t^{a_m} x_{r_{m-1}} t^{a_{m-1}} x_{r_{m-2}} \dots t^{a_1} x_{r_0} t^{a_0} \quad (3.13)$$

where $r_i \in \Phi^+$ and x_{r_i} occurs (r_i, ξ) times in this expression.

Given a word S and $w \in W$, we denote by $w(S)$ the word obtained by substituting everywhere $t^{w(a)}$ and $x_{w(r)}$ for t^a and x_r in the expression of S . In general, $w(S) \neq wSw^{-1}$ because there can be some x_{r_i} in the expression of S for which $r_i \in \Phi^+$ and $w(r_i) \in \Phi^-$. The main point of the following proof is to bring us back to a situation where the equality applies.

i) *Proof of the first relation (3.11):*

We consider the two words $x_r S_\xi$ and $w_r(S_\xi)x_r$. Both words intertwine E_0 with E_ξ and after identifying the two spaces, they coincide with the Weyl transformation w defined by its action on the cell C defined by (2.15): $w(C) = w_r(C) + \xi = w_r(C + \xi)$. Thus they are equal and one has:

$$w_r g_r S_\xi = w_r(S_\xi) w_r g_r \quad (3.14)$$

Now, since $(\xi, r) = 0$ and S_ξ is reduced, S_ξ contains only operators x_{r_i} with $r_i \neq r$ and $w_r(r_i)$ are all in Φ^+ . Thus, $w_r(S_\xi) = w_r S_\xi w_r$. Substituting this equality in (3.14), i) follows.

ii) *Proof of the second relation (3.11):*

Since $(\xi, r) = 1$, one can write S_ξ in the form: $S_\xi = S' x_r$. S' is a word of length $l(S') = l(S_\xi) - 1$ and x_r does not appear in the expression of S' . Using the geometrical interpretation, one shows that:

$$x_{-r} w_r(S') = S_{w_r(\xi)} \quad (3.15)$$

Substituting $w_r(S') = w_r S' w_r$, $x_{-r} = g_r^{-1} w_r$ and $S' = S x_{-r}$ in (3.15) ii) follows.

If ξ is not a dominant weight, we can always write $\xi = \xi_1 - \xi_2$ where ξ_1 and ξ_2 are dominant weights such that: $(\xi_1, r) = 0$ and $(\xi_2, r) = 0$ in case i); $(\xi_1, r) = 1$ and $(\xi_2, r) = 0$ in case ii). Writing $S_\xi = S_{\xi_1} S_{\xi_2}^{-1}$ we obtain the general result.

3.3. Another presentation of the affine-Hecke algebra

Consider the affine generator given by:

$$g_{-r_m+\delta} = t^{-r_m/2} g_{-r_m} t^{r_m/2} \quad (3.16)$$

We could have considered the algebra generated by the subset g_r with $r \in \tilde{\Pi}$ defined by the relations (3.2) (3.3), replacing the non affine root system by the affine root system $\tilde{\Pi}$. Commuting the Weyl group elements w_r to the left of the word $S_{\tilde{w}}$ and reorganizing the translations t^a so as to make only the generators g_r with $r \in \tilde{\Pi}$ appear to the right we obtain the following expression:

$$S_{\tilde{w}} = w^{-1} g_{\tilde{w}} \quad (3.17)$$

where $g_{\tilde{w}}$ is in the subalgebra generated by the $g_r, r \in \tilde{\Pi}$ and w is the projection of \tilde{w} in the Weyl group.

In the general case, we obtain the following expression:

$$S_{\tilde{w},p} = t^{-\gamma} w^{-1} g_{\tilde{w}} \quad (3.18)$$

γ is defined in (2.19). In particular:

$$S_\xi = t^{-\gamma} w_\xi^{-1} g_{\tilde{w}_\xi} \quad (3.19)$$

in this case γ is a minuscule weight. The left-hand side operator $t^{-\gamma} w_\xi^{-1}$ depends only of γ , the projection of ξ in P^V/Q^V . These operators can be added to the generators g_r with $r \in \tilde{\Pi}$ to give a presentation of the affine Hecke algebra by generators and relations.

3.4. representation of the Hecke-algebra in a polynomial space

In this section, we construct a representation of the scattering matrix (2.18) in a space of polynomials in several variables. If we use the expression (3.7) for the operators x_r , all we need is a representation of the operators g_r , $r \in \Pi$, such that the Hecke-algebra relations (3.1), (3.2), (3.3) are satisfied. We also need a representation of the operators t^a satisfying (3.8). Such a representation of the Hecke algebra is known in the mathematical literature (Lusztig, Lascoux and Schutzenberger). Our aim here is to deduce it from the Yang-Baxter equation. In doing so, we shall discover two different sets of operators which obey the affine-Hecke relations: One is given by the S_ξ , the other by the polynomials on which the g_r act.

To construct the representation of the g_r , We recall that the group algebra of the Weight lattice is denoted P . There is a natural action of the Weyl group on P given by:

$$se^\lambda = e^{s(\lambda)}s \quad (3.20)$$

We consider the Hecke algebra (3.4) generated by f^r , $r \in \Pi$. Here the f^r commute with the Weyl group action and the multiplication by e^λ .

We define the operators y_r as follows:

$$y_r = s_r \frac{e^r f_r - f_r^{-1}}{e^r q_r - q_r^{-1}} \quad (3.21)$$

It is easy to show that the Yang-Baxter equations (2.10) (2.11) are equivalent to the fact that y_r with $r \in \Pi$ obey the defining relations of the generators of the Weyl group:

$$y_r^2 = 1 \quad (3.22)$$

$$(y_r y_s)^{m_{rs}} = 1 \quad (3.23)$$

where m_{rs} is the order of $w_r w_s$. It suffices for that to write $x_r = w_r z_r$ in (3.7) and to commute the Weyl group elements w_r to the left of the expressions (2.10),(2.11). If one does the same by commuting s_r to the left of (3.22),(3.23), the identities to verify are the same in both cases.

Let us now quotient the group algebra: $\{\sum c_{\lambda,w,w'} e^\lambda s_w f_{w'}\}$ by the relation $y_r = 1$ to the right. We denote by $\pi(\cdot)$ the projection which consists in eliminating the reflections s_r to the right of an expression in the quotient. This operation eliminates the s_r in the following way:

$$\pi(\dots s_r) = \pi(\dots) \frac{e^r f_r - f_r^{-1}}{e^r q_r - q_r^{-1}} \quad (3.24)$$

The consistency of this projection is assured by the relations (3.22),(3.23). Alternatively, we can define the projection π as follows:

$$\pi(\dots (q_r s_r + (q_r - q_r^{-1}) \frac{1}{e^r - 1} (s_r - 1))) = \pi(\dots) f_r \quad (3.25)$$

Let us rewrite the above relation as $\pi(\dots g_r) = \pi(\dots) f_r$. The consistency of this operation is assured by the fact that the g_r obey the Hecke algebra relations. The expression for g_r is given by:

$$g_r = q_r s_r + (q_r - q_r^{-1}) \frac{1}{e^r - 1} (s_r - 1) \quad (3.26)$$

and the expression (3.12) of x_r is:

$$x_r = q_r + (q_r - q_r^{-1}) \frac{1}{e^{-r} - 1} (1 - s_r) \quad (3.27)$$

One can verify that this representation of the g_r for $r \in \Pi$ and the translations e^λ obey the defining relations (3.11) of a affine Hecke algebra. In fact, it is easy to see that the relations (3.11) are equivalent to the following relations:

$$g_r Q = s_r(Q) g_r + (q_r - q_r^{-1}) \frac{1}{e^r - 1} (s_r(Q) - Q) \quad (3.28)$$

where Q is in the group algebra of the weight lattice. So the above representation is simply obtained by considering the action of g_r on the group algebra P which satisfy the affine Hecke relations with g_r and setting it equal to q_r to the right of an expression.

One important consequence of (3.28) is that any polynomial in P which is Weyl invariant ($s_r(Q) = Q$ for all $r \in \Pi$) is in the center of the affine Hecke algebra: $[g_r, Q] = 0$ for all $r \in \Pi$.

Finally, in order to obtain a representation of the operators S_ξ , we must give the realization of the operators t^a which satisfies the relations (3.8). The operators t^a obey the following commutation relations with P :

$$t^a e^\lambda = e^{(\lambda, a)} e^\lambda t^a \quad (3.29)$$

remarks:

In (3.26), g_r is expressed in terms of the reflection s_r , alternatively, we can express the reflections s_r in terms of the generators g_r . We obtain an expression very similar to (3.21):

$$s_r = \frac{e^r g_r - g_r^{-1}}{e^r q_r - q_r^{-1}} \quad (3.30)$$

but now, the Weyl group relations (3.22),(3.23) satisfied by s_r rely on the fact that e^r and g_r obey the defining relations of the affine Hecke algebra.

In this representation, there are two sets of operators which obey the affine Hecke relations with the operators g_r . One is the group algebra P generated by the spectral parameters e^ξ and the other is the group algebra generated by the S_ξ computed as in (3.13).

4. Affine Hecke algebra, quadratic algebras and physical models

Our aim is to relate the Affine-Hecke relations to the quadratic algebras. In this section we are concerned with specific representations of the quadratic algebras which use the x_r . We show that the spectral parameters which enter the definition of x_r in (3.7) can be taken to obey the affine-Hecke relations (3.11). By considering the operators which commute with the quadratic algebras, one obtains a commuting set of Hamiltonians which describe physical models.

In the first part, we give a brief description of the quadratic algebras which we need in the following. In the second part we describe the procedure of quantization of the spectral parameters of the monodromy matrices which obey the quadratic relations. Finally, in the fourth part, we give a brief description of the physical models which result from this construction.

4.1. Quadratic algebras

Let us consider the quadratic relations for the matrices T_r :

$$x_{e_a - e_b} T_{e_a} T_{e_b} = T_{e_b} T_{e_a} x_{e_a - e_b} \quad (4.1)$$

and

$$x_{e_a - e_b} T_{e_a} x_{e_a + e_b} T_{e_b} = T_{e_b} x_{e_a + e_b} T_{e_a} x_{e_a - e_b} \quad (4.2)$$

Note that these quadratic algebras are obtained from the Yang-Baxter equations (2.11) with $m_{r,s} = 3, 4$ by replacing two of the generators by operators called "Monodromy matrices". The monodromy matrix T_{e_a} can be expanded in the parameter e^a called its spectral parameter. A realization of the relations (4.1)–(4.2) is respectively given by:

$$T_{e_a} = x_{e_a - e_n} x_{e_a - e_{n-1}} \dots x_{e_a - e_1} \quad (4.3)$$

$$T_{e_a} = x_{e_a - e_n} x_{e_a - e_{n-1}} \dots x_{e_a - e_1} x_{e_a} x_{e_a + e_1} \dots x_{e_a + e_n} \quad (4.4)$$

The vectors $e_a, e_b, e_1, \dots, e_n$ form an orthogonal basis. We denote V the vector space spanned by the e_i , $i = 1, \dots, n$. e_a, e_b are called auxiliary vectors. The relations (4.1), (4.2) result from the Yang-Baxter equations (2.11) satisfied by the x_r . In what follows, we consider the representation (3.7) of the operators x_r in terms of generators f_r of the Hecke algebra and w_r of the Weyl group.

$$x_r = w_r \frac{e^r f_r - f_r^{-1}}{e^r q_r - q_r^{-1}} \quad (4.5)$$

We recall that in this formula, the reflection w_r acts on the Hecke generator f_r and that the spectral parameter e^r commutes with f_r and w_r .

4.2. Quadratic algebras and affine Hecke algebras

The Weyl group of A_n (S_n), acts by permuting the vectors e_i of V in the first case (4.1) and The Weyl group of B_n, C_n acts by permuting the vectors and taking their opposite in the second case (4.2). In each case, we consider the fundamental systems given by:

$$\Pi = \{e_1 - e_2, \dots, e_{n-1} - e_n\} \quad (4.6)$$

$$\Pi = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\} \quad (4.7)$$

The generators f_r with $r \in \Pi$ generate a Hecke algebra. Let us consider another realization of the Hecke algebra (3.4) generated by g_r with $r \in \Pi$. The g_r commute with the w_r and the f_r .

We quotient the group algebra of the g_r and the f_r by the relation $f_r = g_r$ to the right. We denote $\pi(\cdot)$ the operation which consists in eliminating the generators g_r in the quotient: $\pi(\dots g_r) = \pi(\dots) f_r$. We now require that the quadratic relations (4.3),(4.4), are still satisfied when one replaces the monodromy matrices T_{e_a} by their projection $\pi(T_{e_a})$. Let us show that the relations are satisfied if the spectral parameters e^r obey the defining relations (3.11) of the affine Hecke algebra with the generators g_r .

In order for the quadratic relations (4.1) (4.2) to be satisfied after taking the quotient The condition to satisfy is:

$$\pi(g_r T_a) = f_r \pi(T_a) \quad \text{for } r \in \Pi \quad (4.8)$$

It ensures that $\pi(T_a T_b) = \pi(T_a) \pi(T_b)$ and therefore that the relations (4.1) (4.2) are satisfied when one substitutes $\pi(T_a)$ for T_a in them.

To verify (4.8) one commutes the generators g_r , $r \in \Pi$ through T_a using the affine Hecke relations with the spectral parameters which enters the definition of the x_r . Once g_r has been pushed to the right, one replaces it with f_r and one commutes f_r back to the left using the Hecke relations and the fact that the Weyl reflections w_r act on the f_r . The computation is simplified by the fact that the denominator of T_a is a Weyl invariant polynomial in P . It therefore commutes with the g_r and one can keep only the numerator of the x_r , ($x_r = w_r(e^r f_r - f_r^{-1})$).

4.3. Physical models

To simplify the discussion, we restrict ourselves to the A_{n-1} case (4.1). We consider n particles with coordinates θ_j on the unit circle. Each particle carries a spin denoted σ_j . They are described by a wave function $\Psi(z_j)$ which is supposed to be polynomial in the variables $z_j = e^{i\theta_j}$.

There is an action of the Hecke algebra on the coordinates which follows from (3.26):

$$g_{j,j+1} = qs_{j,j+1} + (q - q^{-1}) \frac{z_{j+1}}{z_j - z_{j+1}} (s_{j,j+1} - 1) \quad (4.9)$$

where the permutation $s_{j,k}$ permutes the coordinates z_j and z_k . There is also a representation of the Hecke algebra on the spin variables denoted $f_{j,j+1}$ ($f_{j,j+1}$ is supposed to act on the spin of the particles j and $j+1$).

The operators $t_i = t^{e_i}$ act by shifting the variable θ_i :

$$t_i \Psi(z_1, \dots, z_i, \dots) = \Psi(z_1, \dots, tz_i, \dots) \quad (4.10)$$

We call "Physical states" those for which the two Hecke algebras act in the same way.

$$g_{j,j+1} \Psi_{\text{phys}} = f_{j,j+1} \Psi_{\text{phys}} \quad (4.11)$$

It is clear that on these states, we can replace the action of the operators $g_{j,j+1}$ by $f_{j,j+1}$. The operation which consists in replacing the generators $g_{j,j+1}$ by $f_{j,j+1}$ coincides with the projection $\pi(\dots g_{j,j+1}) = \pi(\dots) f_{j,j+1}$ described before. An operator O acting on physical states preserves this space if: $\pi(g_{j,j+1} O) = f_{j,j+1} O$.

Let us denote by S_j the Hecke generators S_{e_j} constructed in (2.30). S_j and $g_{k,k+1}$ obey the relations:

$$\begin{aligned} [g_{k,k+1}, S_j] &= 0 \text{ if } j \neq k, k+1 \\ g_{j,j+1} S_j &= S_{j+1} g_{j,j+1}^{-1} \end{aligned} \quad (4.12)$$

We consider the following Hamiltonians acting on physical states:

$$H_l = \sum_{1 \leq i_1 < i_2 \dots < i_l \leq n} S_{i_1} \dots S_{i_l} \quad (4.13)$$

These operators obviously commute with each other. Moreover, since they are symmetric polynomials in the S_j , they commute with $g_{j,j+1} \forall j$. Thus, their action preserves the physical states. In the simple case where $f_{j,j+1} = q$ (or $-q^{-1}$) (the scalar case), the

projection of these Hamiltonians by π coincides with the trigonometric models defined by Ruijenaars. In this case, the Hamiltonians are invariant under the permutations of the coordinates which makes the identification with Ruijenaars models easy to do. Unfortunately, in the general case, this projection is much more difficult to describe explicitly.

One can also construct the monodromy matrix (4.3). The expression of $x_{e_a - e_i}$ being:

$$x_{e_a - e_i} = s_{a,i} \frac{e^u f_{e_a - e_i} - S_i f_{e_a - e_i}^{-1}}{e^u q - S_i q^{-1}} \quad (4.14)$$

Here e^u is the spectral parameter of the monodromy matrix. Because of (4.8), the monodromy matrix also preserves the physical states and it commutes with H_l . This remains true when one replaces the operators by their projection by π . Thus, the algebra defined by the quadratic relations (4.1) is a symmetry algebra for the H_l .

Let us show that the operators S_j constructed in this way can be diagonalized simultaneously and are unitary.

Eigenvalues of the S_j :

We show that the operators S_j are represented by triangular matrices. Let us recall the expression for the S_j :

$$S_j = x_{j-1,j}^{-1} \dots x_{1,j}^{-1} t_j x_{j,n} \dots x_{j,j+1} \quad (4.15)$$

where the operator $x_{i,j}$ takes the limiting form for $i < j$:

$$x_{i,j} = q + (q - q^{-1}) \frac{z_i}{(z_i - z_j)} (s_{i,j} - 1) \quad (4.16)$$

$x_{i,j}$ commutes with $z_i z_j$ and with z_k for $k \neq i, j$. It acts in the following way for on the monomials $z_i^m, z_j^m, m > 0$ and 1:

$$\begin{aligned} x_{i,j} z_i^m &= q^{-1} z_i^m - (q - q^1)(z_i^{m-1} z_j + \dots + z_i z_j^{m-1}) \text{ for } m > 0 \\ x_{i,j} z_j^m &= q z_j^m + (q - q^{-1})(z_i^m + \dots + z_i z_j^{m-1}) \text{ for } m > 0 \\ x_{i,j} z_i^0 &= 0 \end{aligned} \quad (4.17)$$

To a monomial $z_1^{k_1} \dots z_n^{k_n}$ we associate a partition $|k| = (k_{p_1} \geq k_{p_2} \dots \geq k_{p_n})$ where we order the k_j in decreasing order.

Let us consider which new monomials $z_1^{k'_1} \dots z_n^{k'_n}$ can appear when one acts $x_{i,j}$ on this monomial. First, all the k'_l for $l \neq i, j$ are equal to k_l . Then, if the partition of the new monomial is different from $|k|$, it can be obtained from $|k|$ by a sequence of squeezing

operations: $(..., k_i, ..., k_j, ...) \rightarrow (... , k_i - 1, ..., k_j + 1, ...)$ if $k_i > k_j$. $(..., k_j, ..., k_i, ...) \rightarrow (... , k_j - 1, ..., k_i + 1, ...)$ if $k_i < k_j$. Finally if the partition of the new monomial is equal to $|k|$, $k'_i = k_i$ and $k'_j = k_j$ if $k_i > k_j$.

Let us define an order on the monomials by saying that $z_i^{k_i}$ is larger than $z_i^{k'_i}$ if either $|k'|$ is obtained from $|k|$ by a sequence of squeezing operations, or $|k| = |k'|$ and $k'_i - k_i > 0 \forall i$. It follows from the above analyses that the action of S_j on a monomial produces only monomials which are smaller with respect to this order. Thus the eigenvalues of the operators S_j are given by the diagonal elements in the monomial basis.

Given the partition $|k| = (k_1, \dots, k_n)$, the eigenvalues corresponding to the monomials associated to it are all obtained by permutations of the multiplet:

$$(S_j) = (t^{k_j} q^{n+1-2j}) \quad (4.18)$$

If we set $q = t^{\beta/2}$ with β a real parameter, the operators S_j have the physical interpretation of exponentials of momentum operators: $S_j = t^{K_j}$. $K_j = k_j + \beta(\frac{n+1}{2} - j)$ obey a generalized Pauli principle since they must be β apart from each other.

Scalar product

A scalar product can be defined [6], so that the operators $g_{j,j+1}$ and S_j are unitary: $g_{j,j+1}^+ = g_{j,j+1}^{-1}$, $S_j^+ = S_j^{-1}$. For $q = t^{k/2}$, with k an integer, the scalar product is given by:

$$\langle \Psi_1, \Psi_2 \rangle = \int \prod_{i=1}^n d\theta_i \bar{\Psi}_1(z_j) \Psi_2(z_j) C(z_j) \quad (4.19)$$

where the bar symbol stands for the complex conjugation. q, t, z_j are supposed to be complex numbers of modulus 1 and the integration over θ_j keeps the coefficient of z^0 of the integrand.

The measure $C(z_k)$ is given by:

$$C(z_j) = \prod_{i < j} \prod_{l=-k}^{k-1} (\sqrt{t^l z_i / z_j} - \sqrt{t^{-l} z_j / z_i}) \quad (4.20)$$

In the case of the Ruijzenhaars models, many properties of the wave functions Ψ can be obtained, for example their norms. We refer to [7] for a more complete analyses of their properties.

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